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Resummation of thrust distributions in DIS*

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Abstract: We calculate the resummed distributions for the thrust in DIS in the limit $T \to 1$. Two variants of the thrust are considered: that normalised to $Q/2$, and that normalised to the energy in the current hemisphere. The results expanded to second order are compared to predictions from the Monte Carlo programs DISENT and DISASTER++. A prescription is given for matching the resummed expressions with the full fixed order calculation.

Keywords: QCD, NLO Computations, Deep Inelastic Scattering, Jets.

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1. Introduction

For some time now it has been standard practice in e^+e^- reactions to compare event-shape distributions with resummed perturbative predictions (see for instance [1]). The resummation is necessary because in the two-jet limit (small values of the shape variable) the presence of large logarithms spoils the convergence of the fixed-order calculations. Such resummed analyses have led to valuable information about the strong coupling constant and also about non-perturbative effects [2].

At HERA, similar studies of DIS event-shape distributions are being carried out by both collaborations [3, 4], but as yet no perturbative resummed calculations exist for comparison between data and theory. Here we present the results of such a calculation.

So far resummed predictions for processes involving initial state hadrons exist for observables such as DIS and Drell-Yan cross-sections in the semi-inclusive limits where x for DIS and τ for Drell-Yan are close to one [5, 6, 7]. In addition there are predictions for jet-multiplicities in DIS [8] and the W transverse momentum in Drell-Yan production [9] which entail somewhat similar considerations to those that are encountered here.

In DIS, event shapes are defined in the current hemisphere of the Breit frame to reduce contamination from remnant fragmentation, which is beyond perturbative control [10]. The distribution of partons in the current hemisphere is analogous to that in a single hemisphere in e^+e^-, enabling us to adopt the usual e^+e^- methods for the resummation. But it turns out that event-shape variables also depend on emissions in the remnant hemisphere (typically through recoil effects) — a proper resummed treatment of the space-like branching of the incoming parton is therefore required.

The fact that only one, predefined, hemisphere is used in DIS event shapes leads to a number of subtleties compared to e^+e^- event shapes. Event shapes measure properties of the energy-momentum flow in the current hemisphere. They are always dimensionless, so it is necessary to normalise this measure of energy-momentum flow. Two choices are possible: a normalisation to $Q/2$, or to E, the energy in the current hemisphere (whereas in e^+e^- the total energy E is equal to Q). Furthermore, one often requires the choice of an axis along which to project the momentum flow (for example for the thrust, the broadening). In e^+e^- there is one logical choice of axis, namely the thrust axis (that which maximises the thrust). In DIS, two axes arise naturally: the photon axis and the thrust axis (which is given by the direction of the sum of all momenta in the current hemisphere).

In this article we shall consider two variants of the thrust. Both are measured with respect to the photon axis: one, T_Q, is normalised to $Q/2$ and the other, T_E, is normalised to E. In the limit of a $1+1$ jet event both tend to $T=1$. Since we are particularly interested in this limit, it will be convenient to define $\tau = 1-T$ and consider the limit of small τ. While we are generally interested in the differential distribution $\frac{1}{\sigma}\frac{d\sigma}{d\tau}$, it will turn out to be more convenient to look at the integrated event-shape cross section

$$\Sigma(\tau) = \int_0^\tau d\tau' \frac{d\sigma}{d\tau'}\,, \tag{1.1}$$

from which the distribution can be straightforwardly obtained by differentiation.

For small τ one finds that the shape cross-section contains terms $\alpha_s^n \ln^m \tau$, $m \leq 2n$ coming from multiple soft and collinear radiation. The $\ln\tau$ factors compensate the smallness of α_s, hence the need arises for resummation, which leads to a result schematically of the form $\Sigma(\tau) \sim \exp(-G_{12}\alpha_s \ln^2\tau + \cdots)$, with G_{12} some number which will be calculated. From the point of view of classifying the terms that we shall

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calculate, it turns out to be most convenient to consider $\ln \Sigma$. This contains terms $\alpha_s^n \ln^m \tau$ with $m \leq n+1$. Terms with $m = n+1$ are referred to as leading logarithmic (LL), while those with $m = n$ are known as next-to-leading logarithmic (NLL). We shall resum both these sets of terms, and neglect as subleading the remaining terms, $m < n$.

A word of caution is needed about τ_E, and in general about all event-shape measures which are normalised to the energy in the current hemisphere, E. At $\mathcal{O}(\alpha_s)$ there are configurations in which the current hemisphere is empty. At higher orders it can be filled by soft-gluon radiation from the partons in the remnant hemisphere. If the event-shape is normalised to E then it can take on finite (significantly different from 0 and/or 1) values due to the presence of soft-radiation; in other words it will be infrared unsafe starting from $\mathcal{O}(\alpha_s^2)$. This has been known for some time. The standard experimental practice is to exclude events in which the energy in the current hemisphere is too low. For example, H1 set the threshold $E > \mathcal{E}_{\rm lim} = Q/10$ [3]. Though formally this eliminates any infrared unsafety, one is still left with potentially large logs $\alpha_s(\alpha_s \ln Q/2\mathcal{E}_{\rm lim})^n$. Thus we recommend that $\mathcal{E}_{\rm lim}$ be chosen somewhat larger, say $\mathcal{E}_{\rm lim} = Q/4$. Though this will slightly reduce the statistics, it ought to improve the convergence of the perturbation series.

The structure of this article is as follows. In section 2 we discuss the kinematics and define the thrust measures which will be studied. In section 3 we perform the resummations. Section 4 gives the expressions to be used for the practical calculation of resummed distributions. The results are expanded to second order and compared to the predictions from the fixed order Monte Carlo programs DISASTER++ [11] and DISENT [12]. This enables us to cast some light on the differences between DISASTER++ and DISENT, observed in [13]. Finally a *matching* prescription is proposed for combining the resummed and fixed order predictions. This is followed by the conclusions.

The appendices provide details about the leading order calculation of the thrust (appendix A), the techniques used for the inverse Mellin transforms (appendix B) and the integrations needed for a result correct to NLL (appendix C).

2. Thrust definitions and kinematics

It is convenient to write the momenta k_i of radiated partons (gluons and/or quark-antiquark pairs) in terms of Sudakov (light-cone) variables as (figure 1)

$$k_i = \alpha_i P + \beta_i P' + k_{ti}\,, \qquad \alpha_i \beta_i = \frac{\vec{k}_{ti}^2}{Q^2}\,, \tag{2.1}$$

where P and P' are light-like vectors along the incoming parton and current directions, respectively, in the Breit frame of reference:

$$P = xP_p\,, \qquad P' = xP_p + q\,,$$
$$2(P \cdot P') = -q^2 \equiv Q^2 \qquad (2.2)$$

where P_p is the incoming proton momentum and $x = Q^2/(2P_p \cdot q)$ is the Bjorken variable. Thus in the Breit frame we can write $P = \frac{1}{2}Q(1,0,0,-1)$ and $P' = \frac{1}{2}Q(1,0,0,1)$, taking the current direction as the z-axis. Particles in the *current hemisphere* $\mathcal{H}_\mathrm{C}$ have $\beta_i > \alpha_i$ while those in the proton *remnant hemisphere* $\mathcal{H}_\mathrm{R}$ have $\alpha_i > \beta_i$.

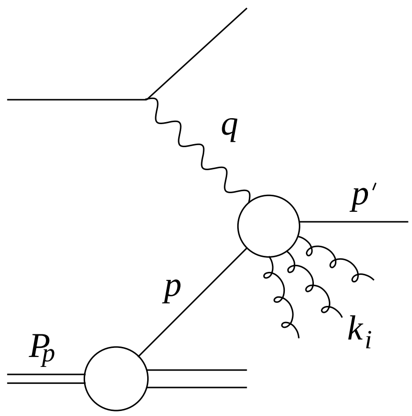

Figure 1: Kinematics.

The momenta p and p' of the initial- and final-state quarks can also be resolved along the Sudakov vectors P and P'. From momentum conservation,

$$p = \left(1 + \alpha' + \sum \alpha_i\right) P\,,$$
$$p' = \alpha' P + \left(1 - \sum \beta_i\right) P' - \sum k_{ti}\,, \qquad (2.3)$$

where

$$\alpha' \left(1 - \sum \beta_i\right) = \frac{\left(\sum \vec{k}_{ti}\right)^2}{Q^2}\,. \qquad (2.4)$$

The initial quark is assumed to be collinear with the proton direction and is therefore aligned along P (neglecting possible 'intrinsic' transverse momentum).

2.1 Thrust normalised to $Q/2$, T_Q

Consider first the thrust T_Q, defined by the sum of longitudinal momenta in the current hemisphere normalised to $Q/2$:

$$T_Q = \frac{2}{Q} \sum_{a \in \mathcal{H}_\mathrm{C}} p_{za} = \frac{2}{Q} \left(p'_z + \sum_{i \in \mathcal{H}_\mathrm{C}} k_{zi} \right), \qquad (2.5)$$

where the index a run over all partons and the index i runs only over emitted partons. We have assumed that the outgoing quark momentum p' lies in the current hemisphere, because the consideration of the non-soft, non-collinear emissions that would result in this not being true is beyond our concern in the present case. We have

$$\tau_Q \equiv 1 - T_Q = \sum \beta_i - \sum_{\mathcal{H}_\mathrm{C}} (\beta_i - \alpha_i) + \alpha'$$
$$= \sum_{\mathcal{H}_\mathrm{R}} \beta_i + \sum_{\mathcal{H}_\mathrm{C}} \alpha_i + \alpha' = \sum \min\{\alpha_i, \beta_i\} + \alpha'\,. \qquad (2.6)$$

Where not explicitly stated, the sums extend over all emitted partons, irrespective of which hemisphere they are in. Here the contribution from those partons that are not directly in the current hemisphere arises through the recoil effect that they have on the current quark.

2.2 Thrust normalised to E, T_E

Another way to define the current jet thrust is to normalise it to the total energy in the current hemisphere, instead of $Q/2$. This gives

$$T_E = \frac{T_Q}{(1-\varepsilon)}\,, \tag{2.7}$$

where

$$\varepsilon \equiv 1 - \frac{2}{Q}\sum_{a\in\mathcal{H}_\mathrm{C}} E_a\,. \tag{2.8}$$

The quantity ε, called the energy deficit in the current hemisphere, is itself an interesting shape variable [14]. It is given by

$$\varepsilon = \sum \beta_i - \sum_{\mathcal{H}_\mathrm{C}}(\beta_i+\alpha_i) - \alpha' = \sum_{\mathcal{H}_\mathrm{R}}\beta_i - \sum_{\mathcal{H}_\mathrm{C}}\alpha_i - \alpha'\,. \tag{2.9}$$

Hence, we have,

$$\tau_E = 1 - T_E = \frac{2}{1-\sum_{\mathcal{H}_\mathrm{R}}\beta_i + \sum_{\mathcal{H}_\mathrm{C}}\alpha_i + \alpha'}\left(\sum_{\mathcal{H}_\mathrm{C}}\alpha_i + \alpha'\right). \tag{2.10}$$

3. Resummation

We shall consider a reduced integrated cross section, where by reduced we mean that it will be the contribution to F_2 (rather than to $d\sigma/dxdQ^2$) which comes from configurations with $1-T<\tau$. There will be no need to consider contributions to F_L since they are suppressed by powers of τ for small τ.

We write the following expression for the (reduced) cross section for the emission of m gluons in the remnant hemisphere and n in the current hemisphere, where all the gluons have $k_t \ll Q$,

$$e_q^2 q_N(Q_0^2)\cdot\frac{1}{m!}\prod_i^m\int_{Q_0^2}^{Q^2}\frac{dk_{ti}^2}{k_{ti}^2}\frac{\alpha_\mathrm{s}(k_{ti}^2)C_F}{2\pi}\rho_{R,i}\cdot\frac{1}{n!}\prod_j^n\int^{Q^2}\frac{dk_{tj}^2}{k_{tj}^2}\frac{\alpha_\mathrm{s}(k_{tj}^2)C_F}{2\pi}\rho_{C,j}\,. \tag{3.1}$$

with

$$\rho_{R,i} = \int z_i^N\, dz_i\,\frac{1+z_i^2}{1-z_i}\,\Theta(Q\alpha_i - k_{ti})\,, \tag{3.2a}$$

$$\rho_{C,j} = \int \frac{d\beta_j}{\beta_j}\left(1+(1-\beta_j)^2\right)\Theta(Q\beta_j - k_{tj})\,. \tag{3.2b}$$

N is the moment variable (conjugate to Bjorken-x) and

$$\alpha_i = \frac{1-z_i}{z_1 \cdots z_i}\,. \tag{3.3}$$

We have introduced a factorisation scale Q_0^2, and accordingly in $\mathcal{H}_\mathrm{R}$ consider only emissions above that scale. Q_0^2 is chosen so as to be much smaller than any other scale in the problem.

For the event shapes considered here, the independent gluon emission pattern written above is enough to reproduce the behaviour of the full QCD matrix elements up to the required (NLL) accuracy, with the following restrictions: α_s must be evaluated to two loops in the CMW (or Physical) scheme [7, 15]; we must also put in the virtual corrections and take into account all possible branchings of the incoming leg, not just the emission of gluons from a quark; but these can both be done trivially later on. It is not necessary to take into account the splitting of emitted partons, other than what is already implicitly included through the running of the coupling.

The fundamental difference between the expression given above and the corresponding one for e^+e^- comes from the presence of the quark distribution. Hard collinear emissions change the momentum fraction of the incoming parton, and this must be accounted for, through a (multiple) convolution of some function (to be determined) with the parton distributions. We take moments (N) with respect to x in order convert that convolution into a product. These moments enter only for the emissions in $\mathcal{H}_\mathrm{R}$ — emissions in $\mathcal{H}_\mathrm{C}$ are essentially identical to those in a hemisphere of e^+e^-.

The use of moments does not however remove all difficulties. In particular in eq. (3.3) we are still left with a product of z_i's. This matters both in the Θ function in eq. (3.2a) and in the contribution to the thrust, which for say τ_Q will be $\beta_i = k_{ti}^2/(Q^2\alpha_i)$ (cf. eq. (2.6)). The solution relies on the fact that the gluons which contribute to the thrust are either the gluon with the highest transverse momentum or soft gluons. Considering the gluons as ordered in transverse momentum, the gluon with highest transverse momentum is number 1 and $\alpha_1 = (1-z_1)/z_1$. One can equally well consider the gluons to be ordered in angle, and if gluon m has the highest transverse momentum, then gluons with $i < m$ must have $1 - z_i \ll 1$ and one can write $\alpha_m \simeq (1-z_m)/z_m$. For soft gluons we can approximate the splitting function by $2/(1-z_i)$, and then replace $(1-z_i)/(z_1 \ldots z_{i-1}) \Rightarrow (1-\zeta_i)$. The integration measure remains $2d\zeta_i/(1-\zeta_i)$, and α_i just becomes $(1-\zeta_i)/\zeta_i$. So we can in general just replace α_i by $(1-z_i)/z_i$ (where z_i should sometimes be understood to mean ζ_i).

3.1 Resummation of τ_Q

To write a resummed expression we take the Mellin transform of our Θ function for τ_Q:

$$\Theta\left(\tau - \sum_{\mathcal{H}_\mathrm{R}} \beta_j - \sum_{\mathcal{H}_\mathrm{C}} \alpha_i - \alpha'\right) = \int \frac{d\nu}{2\pi i\nu} e^{\tau\nu} \left(\prod_{\mathcal{H}_\mathrm{R}} e^{-\nu\beta_i}\right)\left(\prod_{\mathcal{H}_\mathrm{C}} e^{-\nu\alpha_i}\right) e^{-\nu\alpha'}\,. \tag{3.4}$$

This almost factorises the expression for the Θ function into pieces which each depend on a single emission. There remains the less trivial term $e^{-\nu\alpha'}$: from (2.4) we know that α' is approximately (using $1-\sum\beta \simeq 1$) the squared vector sum of emitted transverse momenta. For now it is actually sufficient to consider the sum of squared momenta. For soft and collinear gluons $k_{ti}^2/Q^2 = \alpha_i\beta_i$ is negligible compared to $\min(\alpha_i,\beta_i)$. Only for collinear but hard gluons is k_{ti}^2/Q^2 comparable to $\min(\alpha_i,\beta_i)$ because one of the α_i,β_i is $\mathcal{O}(1)$. In the current hemisphere, since $\beta_i < 1$, the contribution from the k_t^2/Q^2 can be neglected: the region where it is comparable to α_i is single-logarithmic and its inclusion modifies τ by a numerical factor, i.e. changes $\alpha_s \ln\tau$ to say $\alpha_s \ln 2\tau$, a difference which is NNLL and so negligible. The difference is similarly NNLL in the remnant hemisphere, however the fact that α_i can be much greater than 1 (as large as $1/x$), and hence $k_{ti}^2/Q^2 \gg \beta_i$, means that one should at least keep track of it. Therefore in the remnant hemisphere contribution we retain a contribution to τ of the form k_t^2/Q^2 (keeping track of it more accurately would involve using the squared vector sum — but for τ_Q the error is down by one more logarithm). Thus we replace

$$\left(\prod_{\mathcal{H}_\mathrm{R}} e^{-\nu\beta_i}\right)\left(\prod_{\mathcal{H}_\mathrm{C}} e^{-\nu\alpha_i}\right) e^{-\nu\alpha'} \Rightarrow \left(\prod_{\mathcal{H}_\mathrm{R}} e^{-\nu(\beta_i+k_{ti}^2/Q^2)}\right)\left(\prod_{\mathcal{H}_\mathrm{C}} e^{-\nu\alpha_i}\right). \tag{3.5}$$

We then sum over m and n in (3.1) to give us the following exponentiated form for the integrated thrust distribution

$$\Sigma_N(\tau) = e_q^2 q_N(Q^2)\int \frac{d\nu}{2\pi i\nu}\, e^{\tau\nu}\, e^{-\mathcal{R}_\mathrm{R}(\nu)-\mathcal{R}_\mathrm{C}(\nu)}\,. \tag{3.6}$$

The current hemisphere radiator, $\mathcal{R}_\mathrm{C}$ is

$$\mathcal{R}_\mathrm{C}(\nu) = -\int \frac{dk_t^2}{k_t^2}\frac{\alpha_s(k_t^2)C_F}{2\pi}\int_0^1 d\beta\, \frac{1+(1-\beta)^2}{\beta}\,\Theta(Q\beta - k_t)\left(e^{-\nu\alpha}-1\right), \tag{3.7}$$

where the -1 accounts for virtual corrections (as in [16]). The remnant hemisphere radiator is a little trickier. The attentive reader will have noticed that eq. (3.1) has a leading factor of $q_N(Q_0^2)$, whereas eq. (3.6) has $q_N(Q^2)$. Thus $\mathcal{R}_\mathrm{R}(\nu)$ should contain a (ν-independent) piece to take into account the change in scale of the quark distribution,

$$\ln\frac{q_N(Q^2)}{q_N(Q_0^2)} = \int_{Q_0^2}^{Q^2}\frac{dk_t^2}{k_t^2}\frac{\alpha_s(k_t^2)}{2\pi}\gamma_{qq}(N)\,, \tag{3.8}$$

(where γ_{qq} is the standard quark anomalous dimension) as well as the part arising from the sum over m in eq. (3.1). For now we are still working in a framework involving only gluon emission from a quark, hence the use of only the γ_{qq} part of the anomalous dimension matrix to change the scale of the quark distribution.

Therefore we have

$$\mathcal{R}_{\mathrm{R}}(\nu) = -\int_{Q_0^2}^{Q^2} \frac{dk_t^2}{k_t^2}\frac{\alpha_{\mathrm{S}}(k_t^2)C_F}{2\pi} \times \tag{3.9}$$
$$\times\left(\int_0^1 dz\,\frac{1+z^2}{1-z}\Theta(Q\alpha-k_t)\left(z^N e^{-\nu(\beta+k_t^2/Q^2)}-1\right)-\frac{\gamma_{qq}(N)}{C_F}\right),$$

where as before we have introduced a term -1 to account for virtual corrections. Next, we replace $\alpha = (1-z)/z$. In the Θ function we can neglect the z in the denominator (the Θ function is relevant only for small $1-z$). So we obtain

$$\mathcal{R}_{\mathrm{R}}(\nu) = -\int_{Q_0^2}^{Q^2} \frac{dk_t^2}{k_t^2}\frac{\alpha_{\mathrm{S}}(k_t^2)C_F}{2\pi} \times \tag{3.10}$$
$$\times\left(\int_0^1 dz\,\frac{1+z^2}{1-z}\Theta\left(1-z-\frac{k_t}{Q}\right)\left(z^N e^{-\frac{\nu k_t^2}{(1-z)Q^2}}-1\right)-\frac{\gamma_{qq}(N)}{C_F}\right).$$

We note that in the limit of small z the combination $(\beta+k_t^2/Q^2)$ just reduces to k_t^2/Q^2, i.e. a limit on τ is just equivalent to a limit on the emitted transverse momentum. Then we write

$$\gamma_{qq}(N) = C_F\int_0^1 dz\frac{1+z^2}{1-z}(z^N-1)\Theta\left(1-z-\frac{k_t}{Q}\right)+\mathcal{O}\left(\frac{k_t}{Q}\right), \tag{3.11}$$

to obtain

$$\mathcal{R}_{\mathrm{R}}(\nu) = -\int^{Q^2} \frac{dk_t^2}{k_t^2}\frac{\alpha_{\mathrm{S}}(k_t^2)C_F}{2\pi}\int_0^1 dz\,z^N\frac{1+z^2}{1-z}\Theta\left(1-z-\frac{k_t}{Q}\right)\left(e^{-\frac{\nu k_t^2}{(1-z)Q^2}}-1\right). \tag{3.12}$$

The result no longer depends on our original choice of factorisation scale, Q_0^2, and thus we drop the lower limit on the k_t integral. Our next step is to make the approximation, valid to our accuracy [16],

$$\left(e^{-\frac{\nu k_t^2}{Q^2(1-z)}}-1\right) \simeq -\Theta\left(\bar{\nu}-\frac{Q^2(1-z)}{k_t^2}\right), \tag{3.13}$$

with $\bar{\nu} = \nu e^{\gamma_{\mathrm{E}}}$. The k_t integral can then be divided into two pieces (according to whether the limit on $1-z$ in the Θ-function of (3.13) is above or below 1)

$$\mathcal{R}_{\mathrm{R}}(\nu) = \int_{Q^2\bar{\nu}^{-1}}^{Q^2}\frac{dk_t^2}{k_t^2}\frac{\alpha_{\mathrm{S}}(k_t^2)C_F}{2\pi}\int_0^{1-k_t/Q} dz\,z^N\frac{1+z^2}{1-z}+$$
$$+\int_{Q^2\bar{\nu}^{-2}}^{Q^2\bar{\nu}^{-1}}\frac{dk_t^2}{k_t^2}\frac{\alpha_{\mathrm{S}}(k_t^2)C_F}{2\pi}\int_{1-\bar{\nu}k_t^2/Q^2}^{1-k_t/Q} dz\,\frac{2}{1-z}, \tag{3.14}$$

where in the second term we have made the approximation $1 - z \ll 1$. We express the z integral in the first term as

$$\int_0^1 dz \frac{1+z^2}{1-z}(z^N - 1) + \int_0^{1-k_t/Q} dz \frac{1+z^2}{1-z} = \frac{\gamma_{qq}(N)}{C_F} + 2\ln\frac{Q}{k_t} - \frac{3}{2}, \tag{3.15}$$

where we have neglected terms of order k_t/Q. The z integration in the second term of (3.14) gives $2\ln(\bar{\nu}k_t/Q)$.

The current hemisphere radiator can be evaluated in a similar manner, but without the complications from N dependence. The sum of the radiators is

$$\begin{aligned}\mathcal{R}_\mathrm{R} + \mathcal{R}_\mathrm{C} = &\int_{Q^2\bar{\nu}^{-1}}^{Q^2} \frac{dk_t^2}{k_t^2}\frac{\alpha_\mathrm{s}(k_t^2)C_F}{2\pi}\left(\frac{\gamma_{qq}(N)}{C_F} + 4\ln\frac{Q}{k_t} - 3\right) + \\ &+ \int_{Q^2\bar{\nu}^{-2}}^{Q^2\bar{\nu}^{-1}} \frac{dk_t^2}{k_t^2}\frac{\alpha_\mathrm{s}(k_t^2)C_F}{2\pi} 4\ln\left(\frac{\bar{\nu}k_t}{Q}\right).\end{aligned} \tag{3.16}$$

We see that the quark anomalous dimension $\gamma_{qq}(N)$ appears in our answer. It arises from the restriction of transverse momenta of hard emissions. A consideration of all branchings leads to $\gamma_{qq}(N)$ being replaced by the full anomalous dimension matrix. This is a standard result in a situation where a restriction is imposed on the final-state transverse momentum [9]. The inclusion of all possible splittings leads one, however, to consider also soft gluons emitted from a gluon. Fortunately they can be ignored in the present situation because it can be shown that for such a gluon to contribute to the thrust it would have to be emitted at a larger angle than the last hard emission, and thus, by coherence [17] it is emitted with the colour charge of a quark. To see that the angle really is larger, let us consider two gluons contributing equally to the thrust (in the remnant hemisphere), i.e. with similar β values. The angle of the gluon with respect to the incoming quark is proportional to β/k_t. Gluons with transverse momenta smaller than that of the last hard emission and contributing equally to the thrust are bound therefore to have a larger angle.

Thus our final answer in ν space is

$$\begin{aligned}\widetilde{\Sigma}_N(\nu) \simeq e_q^2 q_N\left(\frac{Q^2}{\bar{\nu}}\right)\exp\Bigg[&-\int_{Q^2\bar{\nu}^{-1}}^{Q^2}\frac{dk_t^2}{k_t^2}\frac{\alpha_\mathrm{s}(k_t^2)C_F}{2\pi}\left(4\ln\frac{Q}{k_t} - 3\right) - \\ &- \int_{Q^2\bar{\nu}^{-2}}^{Q^2\bar{\nu}^{-1}}\frac{dk_t^2}{k_t^2}\frac{\alpha_\mathrm{s}(k_t^2)C_F}{2\pi}4\ln\left(\frac{\bar{\nu}k_t}{Q}\right)\Bigg],\end{aligned} \tag{3.17}$$

where the integral involving the anomalous dimension matrix has been absorbed into a rescaling of the quark structure function. For brevity we rewrite the contents of the square bracket as $-2R(\bar{\nu})$. Note that the above expressions which involve the running coupling must be treated with care owing to the Landau pole in the coupling. Hence one has to ensure that $k_t \gg \Lambda_\mathrm{QCD}$ for the above expressions (and subsequent similar expressions) to be formally valid. The perturbative result we derive here (and indeed in the case of T_E) therefore holds for $\tau \gg \Lambda_{\rm QCD}/Q$, which follows from the fact that the smallest scale in the problem must be considered perturbative, $\tau^2 Q^2 \gg \Lambda^2_{\rm QCD}$. However we shall comment later on how these expressions may be modified to take into account non-perturbative effects that originate from scales in the vicinity of the Landau pole.

The inverse Mellin transform is performed using the techniques of [18], as outlined in appendix B and gives

$$\begin{aligned}\Sigma_N(\tau_Q) &\simeq e_q^2 q_N(\tau_Q Q^2)\frac{1}{\Gamma(1+2R')}\exp\left[-2R\left(\frac{e^{\gamma_{\rm E}}}{\tau_Q}\right)\right] &(3.18)\\ &\simeq e_q^2 q_N(\tau_Q Q^2)\frac{1}{\Gamma(1+2R')}\exp\left[-2R\left(\frac{1}{\tau_Q}\right)-2\gamma_{\rm E}R'\right], &(3.19)\end{aligned}$$

where $R' = R'(1/\tau_Q)$. In the scale of the quark distribution, numerical factors such as $e^{\gamma_{\rm E}}$ have been thrown away since they correspond to NNLL effects. Explicit expressions for R and R' are given in appendix C.

3.2 Resummation of τ_E

Starting from the expression for τ_E, (2.10), we have the Θ-function

$$\Theta\left(\frac{\tau}{2}-\sum_{\mathcal{H}_{\rm C}}\alpha_i-\frac{|\sum_{\mathcal{H}_{\rm R}}\vec{k}_{ti}|^2}{Q^2}\right). \tag{3.20}$$

We have taken the contribution to α' only from transverse momenta in the remnant hemisphere, because all transverse momenta in the current hemisphere contribute much less (except for emissions with $\beta \simeq 1$ and $\alpha \simeq \tau$, but this region is not enhanced by any logs and so can be neglected).

We introduce

$$\vec{p}_t = \sum_{\mathcal{H}_{\rm R}}\vec{k}_{ti}\,, \tag{3.21}$$

and write the Θ function as

$$\int\frac{d\nu}{2\pi i\nu}e^{\tau\nu/2}\left(\prod_{\mathcal{H}_{\rm C}}e^{-\nu\alpha_i}\right)\int\frac{d^2b\,d^2p_t}{(2\pi)^2}e^{i\vec{b}.\vec{p}_t}\left(\prod_{\mathcal{H}_{\rm R}}e^{-i\vec{b}.\vec{k}_{ti}}\right)e^{-\nu p_t^2/Q^2}. \tag{3.22}$$

Analogously to what was done in the previous subsection, we write the integrated cross section as

$$\Sigma_N(\tau_E) = e_q^2 q_N(Q^2)\int\frac{d\nu}{2\pi i\nu}e^{\tau\nu/2}\int\frac{d^2p_t\,d^2b}{(2\pi)^2}e^{-\nu p_t^2/Q^2}e^{i\vec{b}.\vec{p}_t}e^{-\mathcal{R}_{\rm R}(b)-\mathcal{R}_{\rm C}(\nu)}. \tag{3.23}$$

Since the contribution from partons in the current hemisphere is the same as in the τ_Q case (except for the factor of 2 included in the definition of the Mellin transform), $\mathcal{R}_{\rm C}$ is the same as in that case, i.e. $\mathcal{R}_{\rm C}(\nu) = R(\bar{\nu})$.

Arguments analogous to those used for τ_Q, eqs. (3.8) to (3.12), lead us to the following result for the remnant-hemisphere radiator $\mathcal{R}_{\mathrm{R}}(b)$,

$$
\begin{aligned}
\mathcal{R}_{\mathrm{R}}(b) &\simeq -\frac{C_F}{2\pi}\int^{Q^2}\frac{d^2\vec{k}_t}{\pi k_t^2}\,\alpha_{\mathrm{s}}(k_t)\int dz\, z^N\frac{1+z^2}{1-z}\Theta\left(1-z-\frac{k_t}{Q}\right)\left(e^{-i\vec{b}.\vec{k}_t}-1\right)\\
&\simeq \frac{C_F}{2\pi}\int^{Q^2}\frac{dk_t^2}{k_t^2}\,\alpha_{\mathrm{s}}(k_t)\left(\frac{\gamma_{qq}(N)}{C_F}+2\ln\frac{Q}{k_t}-\frac{3}{2}\right)(1-J_0(bk_t))\,. \qquad (3.24)
\end{aligned}
$$

From [16] this integral can be evaluated to the required accuracy by replacing

$$
(1-J_0(bk_t)) \Rightarrow \Theta(\bar{b}k_t-1)\,, \qquad \bar{b}\equiv\frac{e^{\gamma_{\mathrm{E}}}b}{2}\,. \qquad (3.25)
$$

Thus we have

$$
\mathcal{R}_{\mathrm{R}}(b) = R_U(\bar{b}^2Q^2)+\int_{1/\bar{b}^2}^{Q^2}\frac{dk_t^2}{k_t^2}\frac{\alpha_{\mathrm{s}}(k_t^2)}{2\pi}\gamma_{qq}(N)\,, \qquad (3.26)
$$

where $R_U(\nu)$ is defined in appendix C. We now perform the p-integration in (3.23) to get

$$
\begin{aligned}
\widetilde{\Sigma}_N(\nu) &\simeq e_q^2 q_N(Q^2)\int\frac{d^2b}{2\pi}\frac{1}{2\nu}e^{-b^2Q^2/4\nu}\exp\left[-R_U(\bar{b}^2Q^2)-\int_{1/\bar{b}^2}^{Q^2}\frac{dk_t^2}{k_t^2}\gamma_{qq}(N)-R(\bar{\nu})\right]\\
&\simeq e_q^2 q_N\left(\frac{1}{\bar{b}_0^2}\right)\int\frac{d^2b}{2\pi}\frac{1}{2\nu}e^{-b^2Q^2/4\nu}\left(\frac{b^2}{b_0^2}\right)^{-R'_U}e^{-R_U(\bar{b}_0^2)-R(\bar{\nu})}\,, \qquad (3.27)
\end{aligned}
$$

where we have expanded $R_U(\bar{b}^2) = R_U(\bar{b}_0^2)+\ln b^2/b_0^2\, R'_U(\bar{b}_0^2)+\mathcal{O}(\alpha_{\mathrm{s}})$ and substituted γ_{qq} with the full anomalous dimension matrix and used it to change the scale of the quark distribution. The integration over b yields

$$
\widetilde{\Sigma}_N(\nu) \simeq e_q^2 q_N\left(\frac{1}{\bar{b}_0^2}\right)\left(\frac{b_0^2Q^2}{4\nu}\right)^{R'_U}\Gamma\left(1-R'_U\right)e^{-R_U(\bar{b}_0^2Q^2)-R(\bar{\nu})}\,, \qquad (3.28)
$$

For simplicity we then choose $b_0^2Q^2 = 4\nu$, or equivalently $\bar{b}_0^2Q^2 = e^{\gamma_{\mathrm{E}}}\bar{\nu}$.

The inverse Mellin transform with respect to ν is similar to the that for τ_Q, and we obtain as our final answer (remembering that ν was conjugate to $\tau/2$)

$$
\Sigma_N(\tau_E) \simeq e_q^2 q_N(\tau_EQ^2)\frac{\Gamma(1-R'_U)}{\Gamma(1+R'+R'_U)}\exp\left[-R\left(\frac{2e^{\gamma_{\mathrm{E}}}}{\tau_E}\right)-R_U\left(\frac{2e^{2\gamma_{\mathrm{E}}}}{\tau_E}\right)\right] \qquad (3.29)
$$

$$
\begin{aligned}
&\simeq e_q^2 q_N(\tau_EQ^2)\frac{\Gamma(1-R'_U)}{\Gamma(1+R'+R'_U)}\exp\left[-R\left(\frac{1}{\tau_E}\right)-R_U\left(\frac{1}{\tau_E}\right)-\right.\\
&\qquad\left.-(\gamma_{\mathrm{E}}+\ln 2)R'-(2\gamma_{\mathrm{E}}+\ln 2)R'_U\right]. \qquad (3.30)
\end{aligned}
$$

As before, $R' = R'(1/\tau)$.

4. The final result

It is possible to go one step further in the resummation, namely the determination of constant terms at $\mathcal{O}(\alpha_s)$. These are terms independent of τ which essentially multiply the resummed answer. Their function is to compensate for mismatches between the independent gluon emission pattern (as applied to a single gluon) and the actual emission pattern, usually in *corners* of phase space, regions not enhanced by any logs. The corners that can arise are the region of hard, non-collinear emission, and, for the thrust, the region of collinear, non-soft emissions with $k_t^2 \sim \tau Q^2$. There are also other origins for such terms, such as certain approximations made in the expression for the thrust, and contributions from virtual factorisation scheme corrections.

The contribution to F_2 from events with $1 - T < \tau$ is therefore given by the following expression:

$$\Sigma(x, Q^2, \tau) = x\left[\sum_{q,\bar{q}} e_q^2 \left(q(x, \tau Q^2) + \frac{\alpha_s(Q^2)}{2\pi} C_{1q} \otimes q(x, Q^2)\right) + \right. \tag{4.1}$$
$$\left. + \left(\sum_{q,\bar{q}} e_q^2\right) \frac{\alpha_s(Q^2)}{2\pi} C_{1g} \otimes g(x, Q^2)\right] e^{Lg_1(\alpha_s\beta_0 L) + g_2(\alpha_s\beta_0 L)}\,,$$

where $L = \ln 1/\tau$ and the convolutions are defined as

$$C_{1q} \otimes q(x, Q^2) = \int_x^1 \frac{dz}{z}\, C_{1q}(z)\, q\left(\frac{x}{z}, Q^2\right), \tag{4.2}$$

and we have written the non-x dependent leading and subleading logarithms as $Lg_1(\alpha_s\beta_0 L)$ and $g_2(\alpha_s\beta_0 L)$ respectively, whose forms follow from eqns. (3.19) and (3.30) and the expressions for R and R_U in appendix C. Specifically, for τ_Q we have (where it should be noted that the R_1 and R_2 are defined with different arguments from R)

$$g_1(\alpha_s\beta_0 L) = -2R_1(\alpha_s\beta_0 L)\,, \tag{4.3a}$$
$$g_2(\alpha_s\beta_0 L) = -2R_2(\alpha_s\beta_0 L) - 2\gamma_E R' - \ln\Gamma(1 + 2R'), \tag{4.3b}$$

and for τ_E,

$$g_1(\alpha_s\beta_0 L) = -\, R_1(\alpha_s\beta_0 L) - R_{U1}(\alpha_s\beta_0 L), \tag{4.4a}$$
$$g_2(\alpha_s\beta_0 L) = -\, R_2(\alpha_s\beta_0 L) - R_{U2}(\alpha_s\beta_0 L) - (\gamma_E + \ln 2)(R' + R'_U) -$$
$$- \gamma_E R'_U + \ln\frac{\Gamma(1 - R'_U)}{\Gamma(1 + R' + R'_U)}\,. \tag{4.4b}$$

It is useful to have their order by order expansions,

$$Lg_1(\alpha_s\beta_0 L) = \sum_n G_{n,n+1} \left(\frac{\alpha_s}{2\pi}\right)^n L^{n+1}\,, \tag{4.5a}$$
$$g_2(\alpha_s\beta_0 L) = \sum_n G_{n,n} \left(\frac{\alpha_s}{2\pi}\right)^n L^n\,, \tag{4.5b}$$

where the $\mathcal{O}(\alpha_s)$ and $\mathcal{O}(\alpha_s^2)$ coefficients are given in table 1.

	G_{12}	G_{11}	G_{23}	G_{22}
τ_Q	$-2C_F$	$3C_F$	$2\pi\beta_0 G_{12}$	$-\frac{4}{3}\pi^2 C_F^2 + \left(\frac{\pi^2}{3} - \frac{169}{36}\right) C_A C_F + \frac{11}{18} C_F n_f$
τ_E	$-\frac{3}{2}C_F$	$(3 - 3\ln 2)C_F$	$\frac{16\pi}{9}\beta_0 G_{12}$	$-\frac{2}{3}\pi^2 C_F^2 + (\frac{\pi^2}{4} - \frac{17}{6} - \frac{22}{3}\ln 2) C_A C_F +$ $+(\frac{1}{3} + \frac{4}{3}\ln 2) n_f C_F$

Table 1: Coefficients in the resummation.

The expressions for the C_1's are given, in the DIS and $\overline{\text{MS}}$ factorisation schemes, in appendix A. These constant terms have been associated with a scale Q^2, both in α_s and in the parton distributions. Strictly such an identification of the scale is not unique: indeed C_1 can come from pieces with a variety of scales, however the error made in neglecting this fact amounts to terms $\alpha_s^{n+1} L^n$ and thus is beyond our accuracy. A further point worth mentioning relates to the cutoff on the visible energy, $\mathcal{E}_{\text{lim}}$. For both τ_Q and τ_E, C_1 is independent of $\mathcal{E}_{\text{lim}}$, as long as events with less than $E < \mathcal{E}_{\text{lim}}$ in the current hemisphere are given a thrust of 0: this is because a modification of $\mathcal{E}_{\text{lim}}$ just redistributes thrust configurations between some finite value of τ and $\tau = 1$; but the small τ region is not affected. (For certain other variables, such as the thrust measured with respect to the thrust axis and normalised to E, this would not be the case).

4.1 Comparison with NLO computations

A valuable test of the resummation is to expand our answers to second order in α_s and compare the result with fixed order predictions from programs such as DISASTER++ [11] and DISENT [12]. Though agreement does not guarantee the correctness of the resummation, it is nevertheless a non-trivial check.

For the comparison we simply look at the second order contribution to

$$\frac{1}{\sigma_{0,N}(Q^2)} \frac{d\Sigma_N(Q^2,\tau)}{d\ln 1/\tau}, \tag{4.6}$$

where $\sigma_{0,N}$ is just the $\mathcal{O}(\alpha_s^0)$ part of the structure function moment $F_{2,N}(Q^2)$. Moment space with respect to x is used to reduce the various convolutions (for the C_1's and the change of scale of the parton distribution) to products. At order α_s^2, the terms that we control go as $\alpha_s^2 L^3$, $\alpha_s^2 L^2$ and $\alpha_s^2 L$ and their coefficients can be determined by expanding (4.1). Therefore the difference between our expanded resummed answer and the exact second order coefficient is a term which for large L is at most a constant. Figure 2 shows the expanded resummed and exact second order results (from the fixed order Monte Carlo program DISASTER++) in the quark and gluon channels.

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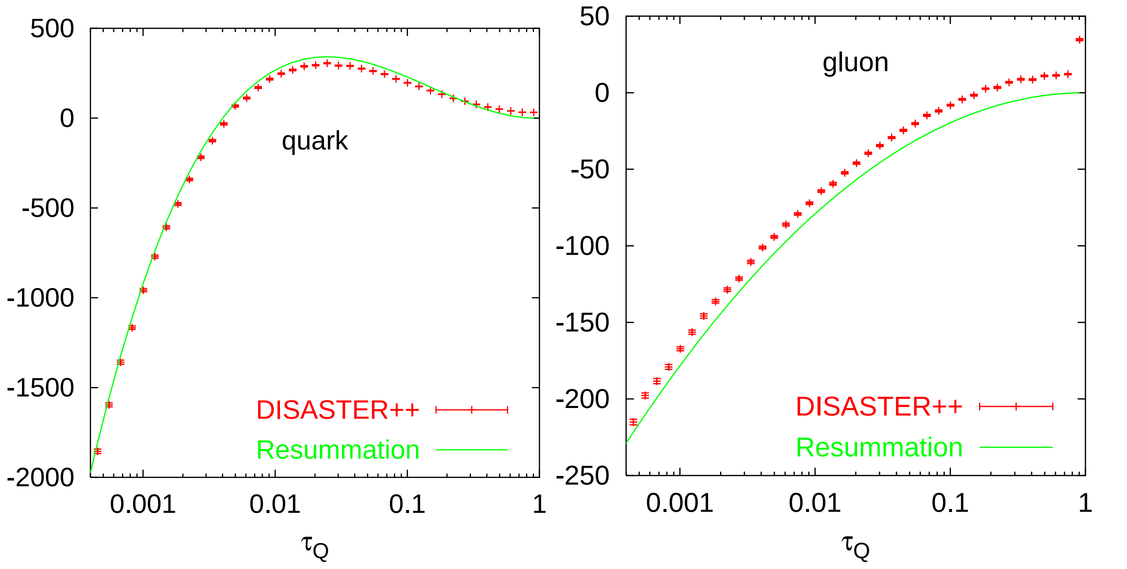

Figure 2: For τ_Q, a comparison of the coefficient of $(\alpha_s/2\pi)^2$ from the expanded resummed answer and that from DISASTER++. Shown for $N = 1.4$.

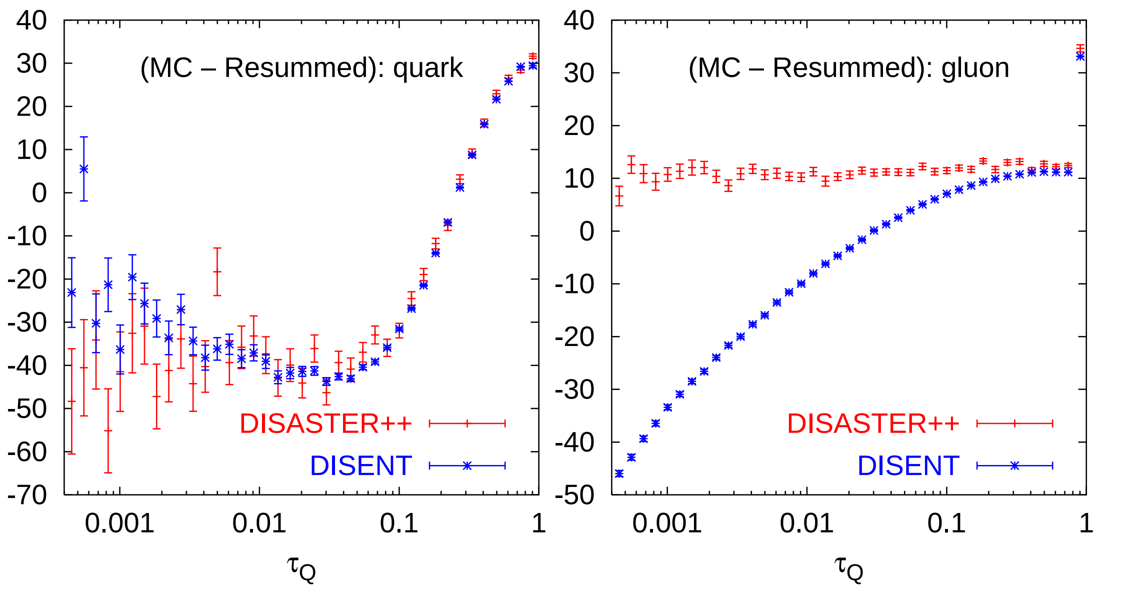

Figure 3: For τ_Q, the difference between the second order coefficient from the Monte Carlo programs (DISASTER and DISENT) and that from the expanded resummed answer. Shown for $N = 1.4$.

The shape of the resummed and exact results are clearly similar and the difference between them does seem to tend to a constant at small τ. This is visible more clearly in figure 3 which shows the difference between the two results. Also shown is the difference between the current version (0.1) of DISENT and the resummed answer. In the gluon channel the difference is clearly inconsistent with a constant: it seems to go as $\alpha_s^2 L^2$. In the quark channel there is also evidence of similar problems, with

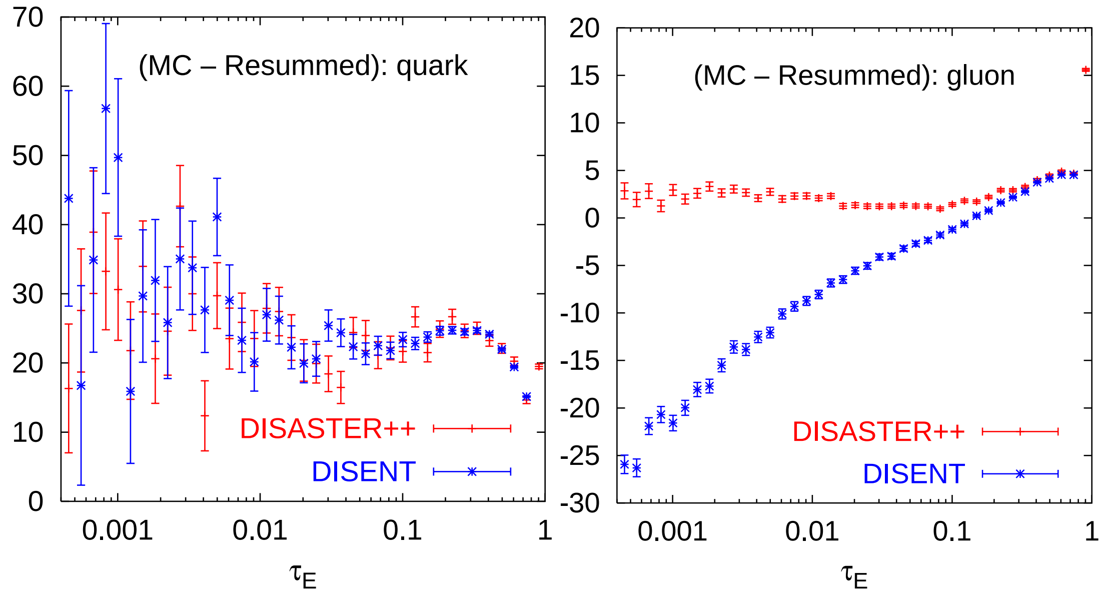

Figure 4: For τ_E, the difference between the second order coefficient from the Monte Carlo programs and that from the expanded resummed answer. Shown for $N = 1.7$.

a significance of a few (between 3 and 5) standard deviations. Results are shown for τ_E in figure 4, where the same pattern of successes and problems is seen.

The origin of the discrepancy in DISENT has yet to be elucidated. However it has been possible to identify that the disagreement lies in pieces proportional to C_F^2 (quark channel) and $C_A T_R$ (gluon channel). Furthermore the fact that, in the gluon channel, the disagreement is at the level of a $\alpha_s^2 L^2$ term in the distribution ($\alpha_s^2 L^3$ in the integrated distribution) suggests that the problem lies with a term involving one soft and two collinear divergences. In the gluon channel there should be *no* term $\alpha_s^2 L^2$ with colour factor $C_A T_R$ (due to angular ordering, soft gluon radiation in the region relevant to the thrust is associated with the colour factor C_F because it is radiated at an angle larger than that of the quark emitted in the photon-gluon fusion).

Above we have said that the difference between the resummed and DISASTER++ is consistent with a constant. This should be qualified: in the gluon channel, the largest-allowed non-constant is quite small, about $\pm 0.2(\alpha_s/2\pi)^2 L$ for both τ_Q and τ_E. In the quark channel the statistical error on the results is much larger, because the largest term goes as $\alpha_s^2 L^3$ (in the gluon channel it goes as $\alpha_s^2 L^2$), and for a given number of events the statistical error is proportional roughly to the largest term. The largest allowed non-constant term here would go as $\sim \pm 3(\alpha_s/2\pi)^2 L$. In both the quark and gluon channels the uncertainty represents about 10% of the coefficient of the $(\alpha_s/2\pi)^2 L$ term from the resummation.

The DISASTER++ results have been obtained[1] with the equivalent of 50 days'

[1] Using the Condor system [19] to distribute the processing across INFN computers spread across

time on a machine with a SPEC CFP95 [20] of about 20 (or roughly equivalently, 500 MIPS). Thus significantly higher precision is not really feasible. (DISENT, for comparable errors, is an order of magnitude faster, so when the problems with DISENT have been resolved a more stringent comparison will be feasible; DISENT also shows less dependence on the internal cutoff, required in order to avoid floating point errors).

4.2 Matching

The above form for the answer is satisfactory for very small τ. However in practice one wishes to fit to data over a range of τ. In e^+e^- sophisticated methods have been developed for combining the $\mathcal{O}(\alpha_s^2)$ and the resummed calculations in order to have an expression which can be applied over the whole range of data [1]. This procedure is usually called matching. It essentially involves two elements: the removal of terms which would otherwise lead to double counting (i.e. logarithms present in both the fixed-order and resummed answers) and the exponentiation of the subleading logarithms ($\alpha_s^2 L$ which is associated with the coefficient G_{21} of $g_3(\alpha_s\beta_0 L)$) which are present in the fixed-order calculation.

In DIS the situation is a little more delicate because the term $\alpha_s^2 L$ which would normally be associated with G_{21} is difficult to exponentiate — for example some of it might arise because the parton distributions convoluted with the C_1's should really have been evaluated at a different scale, or because of contributions from the NLO splitting functions. Some of these contributions exponentiate only in moment space (with respect to x) and as a matrix in flavour space. Thus the simple exponentiation carried out in e^+e^- is not appropriate here.

However such an exponentiation is not strictly necessary. What is necessary [1] is to ensure that terms such as $\alpha_s^2 L$ are not independent of the resummation, but rather a subleading modification of the resummation; thus our final answer must contain not $\alpha_s^2 L$ on its own, but $\alpha_s^2 L \exp(Lg_1(\alpha_s\beta_0 L) + \cdots)$.

So far we have not yet carried out a detailed study of possible matching schemes. However for completeness we do include a preliminary proposal for matching, which satisfies the requirements discussed above and which will be examined in more depth elsewhere [21].

Since these matched expressions will be directly compared to data, it is useful at this point to switch to considering the cross section, differential in x and Q^2, for $1-T$ to be less than τ. For the resummed differential cross section, which we will represent by $d^2\sigma_R$, we have

$$d^2\sigma_R \equiv \frac{d^2\sigma_R(x,Q^2,\tau)}{dx\,dQ^2} = \frac{4\pi\alpha^2}{xQ^4}\frac{(1+(1-y)^2)}{2}\Sigma(x,Q^2,\tau)\,. \qquad (4.7)$$

the whole of Italy

For brevity it will turn out to be useful to define the following matrices:

$$\mathbf{q}(x) = \begin{pmatrix} q_u(x) \\ q_{\bar{u}}(x) \\ \vdots \\ g(x) \end{pmatrix}, \qquad \mathbf{P}(x) = \begin{pmatrix} P_{qq}(x) & 0 & \cdots & P_{qg}(x) \\ 0 & P_{qq}(x) & & \\ \vdots & & \ddots & \\ P_{gq}(x) & & & P_{gg}(x) \end{pmatrix}, \tag{4.8}$$

and

$$\mathbf{C}_0^T(x) = \frac{4\pi\alpha^2}{Q^4}\frac{(1+(1-y)^2)}{2}\begin{pmatrix} e_u^2\delta(1-x) \\ e_u^2\delta(1-x) \\ \vdots \\ 0 \end{pmatrix}, \tag{4.9}$$

$$\mathbf{C}_1^T(x) = \frac{4\pi\alpha^2}{Q^4}\frac{(1+(1-y)^2)}{2}\begin{pmatrix} e_u^2 C_{1,q}(x) \\ e_u^2 C_{1,q}(x) \\ \vdots \\ \sum_{q,\bar{q}} e_q^2 C_{1,g}(x) \end{pmatrix}. \tag{4.10}$$

With this notation, and using eqs. (4.1) and (4.7), the resummed differential cross section becomes

$$d^2\sigma_R(x,Q^2,\tau) = \left[\mathbf{C}_0 \otimes \mathbf{q}(x,\tau Q^2) + \bar{\alpha}_{\mathrm{s}}(Q^2)\mathbf{C}_1 \otimes \mathbf{q}(x,Q^2)\right] e^{Lg_1(\alpha_{\mathrm{s}}\beta_0 L)+g_2(\alpha_{\mathrm{s}}\beta_0 L)}, \tag{4.11}$$

where we have introduced $\bar{\alpha}_{\mathrm{s}} = \alpha_{\mathrm{s}}/2\pi$.

The form that we propose for the matching is the following

$$\begin{aligned} d^2\sigma(x,Q^2,\tau) = \; & d^2\sigma_R + \Big[\bar{\alpha}_{\mathrm{s}}\left(d^2\sigma_E^{(1)} - d^2\sigma_R^{(1)}\right) + \\ & + \bar{\alpha}_{\mathrm{s}}^2\left(d^2\sigma_E^{(2)} - d^2\sigma_R^{(2)} - (d^2\sigma_E^{(1)} - d^2\sigma_R^{(1)})(L^2G_{12} + LG_{11})\right)\Big] e^{Lg_1(\alpha_{\mathrm{s}}\beta_0 L)+g_2(\alpha_{\mathrm{s}}\beta_0 L)}, \end{aligned} \tag{4.12}$$

where $d^2\sigma_E^{(n)}$ is the coefficient of $\bar{\alpha}_{\mathrm{s}}^n$ in the exact result (as determined from Monte Carlo programs) and $d^2\sigma_R^{(n)}$ is the coefficient of $\bar{\alpha}_{\mathrm{s}}^n$ in the resummed answer:

$$d^2\sigma_R^{(1)} = \left[\mathbf{C}_0\left(G_{12}L^2 + G_{11}L\right) - \mathbf{C}_0 \otimes \mathbf{P}L + \mathbf{C}_1\right] \otimes \mathbf{q}(x,Q^2), \tag{4.13}$$

$$\begin{aligned} d^2\sigma_R^{(2)} = \; & \Bigg[\mathbf{C}_0\left(\frac{1}{2}G_{12}^2L^4 + (G_{23} + G_{11}G_{12})L^3 + \left(\frac{1}{2}G_{11}^2 + G_{22}\right)L^2\right) + \\ & + \mathbf{C}_0 \otimes \left(-G_{12}\mathbf{P}L^3 + \left(\frac{1}{2}\mathbf{P}\otimes\mathbf{P} - G_{11}\mathbf{P} - \pi\beta_0\mathbf{P}\right)L^2\right) + \\ & + \mathbf{C}_1(G_{12}L^2 + G_{11}L)\Bigg] \otimes \mathbf{q}(x,Q^2). \end{aligned} \tag{4.14}$$

For these expressions to be correct, in eq. (4.11) the evolution of the parton distributions to scale τQ^2 must be carried out from scale Q^2 with just the leading log DGLAP equations (in any case this is all that can be guaranteed by the resummation procedure). If one wants to use the NLL DGLAP equations, one may, but then eq. (4.14) should be modified accordingly.

Finally we comment on the effect of the emission of gluons with transverse momenta $k_t \approx \Lambda_{\rm QCD}$. Such gluons (gluers) are responsible for the appearance of power corrections which modify the perturbative predictions for shape variables starting at the $1/Q$ level. A detailed study of such effects has been carried out in refs. [22, 23, 27] for e^+e^- and specifically for DIS in [14].

It has been argued in ref. [27], using an operator language, that the effect of gluer emission on event shape distributions is to smear the corresponding perturbative distributions. Formally, in the above approach, one has to perform a convolution of the perturbative spectrum with a variable dependent 'shape-function' which takes account of non-perturbative effects. However in the region where $\tau \gg \Lambda_{\rm QCD}/Q$ it is possible to view the effect of the power correction as a simple shift of the entire perturbative spectrum, which emerges naturally in our approach by applying the dispersive representation of the running coupling in the radiator as is the case in e^+e^- annihilation [22, 23]. The amount of the shift (towards larger τ values) is identical to the power correction to the mean values of the shape variables calculated in ref. [14].[2]

5. Conclusions

We have determined the resummed distributions for two definitions of thrust, in the region $T \to 1$ with the condition $1 - T \gg \Lambda_{\rm QCD}/Q$. For τ_Q the answer is almost identical to the e^+e^- result, except that the answer is multiplied by a parton distribution, which is evaluated not at scale Q^2 but at scale τQ^2. This arises because placing a limit on the thrust amounts, in the remnant collinear region, to placing a limit on the emitted transverse momentum. The thrust normalised to the energy in $\mathcal{H}_{\rm C}$ also has this dependence on the quark distribution at τQ^2. But it differs from τ_Q in that the coefficient of the leading double logarithm, G_{12} is only 3/4 of that of τ_Q, the reason being that the two normalisations depend differently on emissions in the remnant hemisphere.

These resummed expressions have been expanded and compared to the predictions from the fixed-order Monte Carlo programs DISASTER++ and DISENT. We find reasonable agreement with DISASTER++, but a definite discrepancy compared to DISENT. The discrepancy is most visible in the gluon channel, but there is evidence that there might be a problem also in the quark channel.

[2]With the proviso that the Milan factor [22, 23] should take on the updated value of 1.49 for three light flavours [24, 25].

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We have also given a ‘matching’ prescription for combining the resummed and fixed order predictions: it satisfies the basic properties of being correct both to second fixed order and for the leading and sub-leading logarithms, and additionally of not having ‘large’ (constant or logarithmic) pieces left over at small τ.

There remain a certain number of other DIS event-shape variables that can be resummed. They include the broadening, the C-parameter and the thrust measured with respect to the thrust axis and the jet mass. These will be examined elsewhere [21].

Acknowledgments

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A. Leading order calculations

Here we give the explicit expressions for the functions $C_{1q}(z)$ and $C_{1g}(z)$ entering the result, eq. (4.1), for the integrated thrust distribution. Essentially, they are obtained by carrying out the full leading-order calculation, taking the limit of small τ, and then subtracting the pieces with a logarithmic dependence on τ.

We give results for the F_2 part only, in accord with eq. (4.1), since the component of the cross section contributing to F_L is suppressed by a power of τ at small τ. In the DIS factorisation scheme, the results for τ_Q are as follows:

$$C_{1q}^{\mathrm{DIS}}(z) = -C_F\left[\frac{1+z^2}{1-z}\ln\frac{1}{z} + \frac{1+2z-6z^2}{2\,(1-z)_+}\right], \tag{A.1}$$

$$C_{1g}^{\mathrm{DIS}}(z) = -T_R\left[\left(z^2+(1-z)^2\right)\ln\frac{1}{z} - (1-6z\,(1-z))\right]. \tag{A.2}$$

The plus prescription has its usual meaning. The equivalent results in moment space (with respect to z) are

$$C_{1q,N}^{\mathrm{DIS}} = -C_F\left[\frac{5}{N+1} - \frac{1}{N^2\,(N+1)^2} + \frac{3}{2}\,(\psi(N)+\gamma) + 2\psi'(N)\right], \tag{A.3}$$

$$C_{1g,N}^{\mathrm{DIS}} = -T_R\left[\frac{2}{(N+2)^2} - \frac{2}{(N+1)^2} + \frac{1}{N^2} - \frac{1}{N} + \frac{6}{N+1} - \frac{6}{N+2}\right], \tag{A.4}$$

where $\psi(N) = d/dN \ln\Gamma(N)$.

The analogous results for τ_E are somewhat more complicated, because of the more involved nature of the phase space integrations for that variable:

$$C_{1q}^{\mathrm{DIS}}(z) = -\,C_F\left[\frac{1+z^2}{1-z}\ln\frac{1}{z} + (1+z^2)\left(\frac{\ln(2(1-z))}{1-z}\right)_+ + \frac{1+4\ln 2+2z-6z^2}{2\,(1-z)_+} - \right.$$
$$\left. -\,(1+z)\ln 2 + \frac{3}{2}\left(\ln^2 2 - \ln 2\right)\delta(1-z)\right], \tag{A.5}$$

$$C_{1g}^{\mathrm{DIS}}(z) = -\,T_R\left[\left(z^2+(1-z)^2\right)\ln\frac{2(1-z)}{z} + \left(-1+6z-6z^2\right)\right]. \tag{A.6}$$

The results in moment space are

$$C_{1q,N}^{\mathrm{DIS}} = -\,C_F\left[\frac{\pi^2}{6} + \frac{2}{N} + \frac{3}{N+1} + \frac{3}{2}\psi(N) + \frac{3}{2}\gamma + \gamma\psi(N) + \gamma\psi(N+2) + \gamma^2 + \right.$$
$$+\,\frac{1}{2}\left(\psi'(N)+\psi'(N+2)\right) + \frac{1}{2}\left((\psi(N))^2 + (\psi(N+2))^2\right) +$$
$$\left. +\,\frac{3}{2}(\ln^2 2 - \ln 2) - \ln 2\left(2\left(\psi(N)+\gamma\right) + \frac{1}{N} + \frac{1}{N+1}\right)\right], \tag{A.7}$$

$$C_{1g,N}^{\mathrm{DIS}} = -\,T_R\left[-\frac{1}{N}\left(1+\psi(N)+\gamma\right) + \frac{2}{N+1}\left(3+\psi(N+1)+\gamma\right) - \right.$$
$$\left. -\,\frac{2}{N+2}\left(3+\psi(N+2)+\gamma\right) + \ln 2\left(\frac{1}{N} - \frac{2}{N+1} + \frac{2}{N+2}\right)\right]. \tag{A.8}$$

Additionally to go from the DIS to the $\overline{\mathrm{MS}}$ factorisation scheme result, it suffices to add the $\mathcal{O}(\alpha_s)$ $\overline{\mathrm{MS}}$-scheme F_2 coefficient function to the above results:

$$C_{1q}^{\overline{\mathrm{MS}}} = C_{1q}^{\mathrm{DIS}} + C_F\left[2\left(\frac{\ln(1-z)}{1-z}\right)_+ - \frac{3}{2}\frac{1}{(1-z)_+} - (1+z)\ln(1-z) - \right.$$
$$\left. -\,\frac{1+z^2}{1-z}\ln z + 3 + 2z - \left(\frac{\pi^2}{3}+\frac{9}{2}\right)\delta(1-z)\right], \tag{A.9}$$

$$C_{1g}^{\overline{\mathrm{MS}}} = C_{1g}^{\mathrm{DIS}} + T_R\left[\left((1-z)^2+z^2\right)\ln\frac{1-z}{z} - 8z^2 + 8z - 1\right]. \tag{A.10}$$

B. Inverse Mellin transforms

In [18] the following operator technique was developed to help carry out the Mellin transforms. We write

$$e^{-\mathcal{R}(\nu)} = e^{-\mathcal{R}(e^{-\partial_a})}\,\nu^{-a}\Big|_{a=0}. \tag{B.1}$$

The inverse Mellin transform is given by

$$\Sigma(\tau) = \int\frac{d\nu}{2i\pi\nu}e^{\tau\nu}e^{-\mathcal{R}(e^{-\partial_a})}\,\nu^{-a} = e^{-\mathcal{R}(e^{-\partial_a})}\frac{1}{\Gamma(1+a)}\left(\frac{1}{\tau}\right)^{-a}\Bigg|_{a=0}. \tag{B.2}$$

Using the identity

$$e^{-\mathcal{R}(e^{-\partial_a})}x^{-a}g(a)\Big|_{a=0} = e^{-\mathcal{R}(xe^{-\partial_a})}g(a)\Big|_{a=0}, \tag{B.3}$$

we can absorb a power of $(1/\tau)$ into the argument of the radiator

$$\Sigma(\tau) = e^{-\mathcal{R}\left(\frac{1}{\tau}e^{-\partial_a}\right)}\frac{1}{\Gamma(1+a)}\Big|_{a=0}. \tag{B.4}$$

Performing the logarithmic expansion of the radiator:

$$-\mathcal{R}\left(xe^{-\partial_a}\right) = -\mathcal{R}(x) + \mathcal{R}'(x)\partial_a - \frac{1}{2}\mathcal{R}''(x)\partial_a^2 + \cdots, \tag{B.5}$$

we obtain that the action of the operator on a function which is *regular* in the origin reduces to substituting $\mathcal{R}'(x)$ for a, while $\mathcal{R}''(x) = \mathcal{O}(\alpha_s)$ and higher derivatives produce negligible corrections:

$$\Sigma(\tau) = e^{-\mathcal{R}(1/\tau)}\frac{1}{\Gamma(1+\mathcal{R}')}, \qquad \mathcal{R}' = \mathcal{R}'\left(\frac{1}{\tau}\right). \tag{B.6}$$

C. Two-loop radiator integrals

There are two integrals which have to be evaluated:

$$R_U(\nu) = \int_{Q^2\nu^{-1}}^{Q^2}\frac{dk_t^2}{k_t^2}\frac{\alpha_s(k_t^2)C_F}{2\pi}\left(\ln\frac{Q^2}{k_t^2} - \frac{3}{2}\right), \tag{C.1}$$

and

$$R_L(\nu) = \int_{Q^2\nu^{-2}}^{Q^2\nu^{-1}}\frac{dk_t^2}{k_t^2}\frac{\alpha_s(k_t^2)C_F}{2\pi}\ln\left(\frac{\nu^2k_t^2}{Q^2}\right). \tag{C.2}$$

For the results to be correct at two-loop accuracy it is necessary that α_s be the CMW or Physical scheme coupling [7] and that the running of α_s be taken into account to two-loop level. The CMW scheme coupling is related to the $\overline{\text{MS}}$ coupling through

$$\alpha_s^{\text{CMW}} = \alpha_s^{\overline{\text{MS}}}\left(1 + \frac{\alpha_s}{2\pi}K\right), \qquad K = C_A\left(\frac{67}{18} - \frac{\pi^2}{6}\right) - \frac{5}{9}n_f, \tag{C.3}$$

and the two-loop running of the coupling is reproduced to sufficient accuracy by the expression

$$\alpha_s(\mu^2) = \alpha_s(Q^2)\left[\frac{1}{1-\lambda} - \frac{\beta_1}{\beta_0}\frac{\alpha_s(Q^2)\ln(1-\lambda)}{(1-\lambda)^2}\right], \qquad \lambda = \alpha_s(Q^2)\beta_0\ln\frac{Q^2}{\mu^2}, \tag{C.4}$$

where

$$\beta_0 = \frac{11C_A - 2n_f}{12\pi}, \qquad \beta_1 = \frac{17C_A^2 - 5C_An_f - 3C_Fn_f}{24\pi^2}. \tag{C.5}$$

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We split the results into leading and subleading logarithms:

$$R_U(\nu) = \ln\nu \cdot R_{U1}\left(\alpha_s(Q^2)\beta_0 \ln\nu\right) + R_{U2}(\alpha_s\beta_0\ln\nu) \tag{C.6}$$

and analogously for R_L and $R = R_U + R_L$. The results are as follows:

$$\begin{aligned} R_{U1}(\lambda) =& \frac{C_F}{2\pi\beta_0\lambda}\left[-\lambda - \ln(1-\lambda)\right], && \text{(C.7a)}\\ R_{U2}(\lambda) =& \frac{3C_F}{4\pi\beta_0}\ln(1-\lambda) + \frac{KC_F[\lambda + (1-\lambda)\ln(1-\lambda)]}{4\pi^2\beta_0^2(1-\lambda)} + \\ &+ \frac{C_F\beta_1}{2\pi\beta_0^3}\left[-\frac{\lambda+\ln(1-\lambda)}{1-\lambda} - \frac{1}{2}\ln^2(1-\lambda)\right], && \text{(C.7b)} \end{aligned}$$

and

$$\begin{aligned} R_{L1}(\lambda) =& \frac{C_F}{2\pi\beta_0\lambda}\left[\lambda + (1-2\lambda)\ln\left(\frac{1-2\lambda}{1-\lambda}\right)\right], && \text{(C.8a)}\\ R_{L2}(\lambda) =& \frac{KC_F\left[(1-\lambda)\ln\frac{1-\lambda}{1-2\lambda} - \lambda\right]}{4\pi^2\beta_0^2(1-\lambda)} + \frac{C_F\beta_1}{2\pi\beta_0^3}\left[\frac{\lambda + (2\lambda-1)\ln(1-\lambda)}{1-\lambda} + \right. \\ &\left. + \frac{1}{2}\ln^2(1-2\lambda) - \frac{1}{2}\ln^2(1-\lambda) + \ln(1-2\lambda)\right], && \text{(C.8b)} \end{aligned}$$

and for the sum $R = R_U + R_L$,

$$\begin{aligned} R_1(\lambda) =& \frac{C_F}{2\pi\beta_0\lambda}\left[(1-2\lambda)\ln(1-2\lambda) - 2(1-\lambda)\ln(1-\lambda)\right], && \text{(C.9a)}\\ R_2(\lambda) =& \frac{3C_F}{4\pi\beta_0}\ln(1-\lambda) + \frac{KC_F[2\ln(1-\lambda) - \ln(1-2\lambda)]}{4\pi^2\beta_0^2} + && \text{(C.9b)}\\ &+ \frac{C_F\beta_1}{2\pi\beta_0^3}\left[\ln(1-2\lambda) - 2\ln(1-\lambda) + \frac{1}{2}\ln^2(1-2\lambda) - \ln^2(1-\lambda)\right]. \end{aligned}$$

We also need the expressions for the derivatives of the R's with respect to $\ln\nu$:

$$\begin{aligned} R'_U =& \frac{C_F}{2\pi\beta_0}\frac{\lambda}{1-\lambda}, && \text{(C.10a)}\\ R'_L =& \frac{C_F}{2\pi\beta_0}\left(2\ln\frac{1-\lambda}{1-2\lambda} - \frac{\lambda}{1-\lambda}\right), && \text{(C.10b)}\\ R' =& \frac{C_F}{\pi\beta_0}\ln\frac{1-\lambda}{1-2\lambda}, && \text{(C.10c)} \end{aligned}$$

where $\lambda = \alpha_s\beta_0\ln\nu$. A final point relates to scale changes. If in $\ln\nu R_1(\lambda)$ we change the scale of α_s from Q^2 to μ^2, then R_2 gets modified as follows:

$$R_2 \to R_2 + \lambda\ln\frac{\mu^2}{Q^2}\left(R' - R_1\right), \tag{C.11}$$

and analogously for R_U and R_L.

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